烟台理工学院
Yantai Institute of Technology
（原烟台大学文经学院）
(Wenjing College Yantai University)

机器人学

人工智能学院　杨智勇
二零二一年八月二十日

第五章 速度和静力

时变位置和姿态的符号表示

- 位置矢量 P_Q 的微分

$$^{B}V_Q = \frac{d}{dt}\,^{B}P_Q = \lim_{\Delta t \to 0} \frac{^{B}P_Q(t+\Delta t) - \,^{B}P_Q(t)}{\Delta t}$$

位置矢量 $^{B}P_Q$ 相对于坐标系 {B} 的微分

$$^{A}(\,^{B}V_Q) = \,^{A}(\frac{d}{dt}\,^{B}P_Q)$$

表达在坐标系 {A} 中

$$= \,^{A}_{B}R\,^{B}(\,^{B}V_Q) = \,^{A}_{B}R\,^{B}V_Q$$

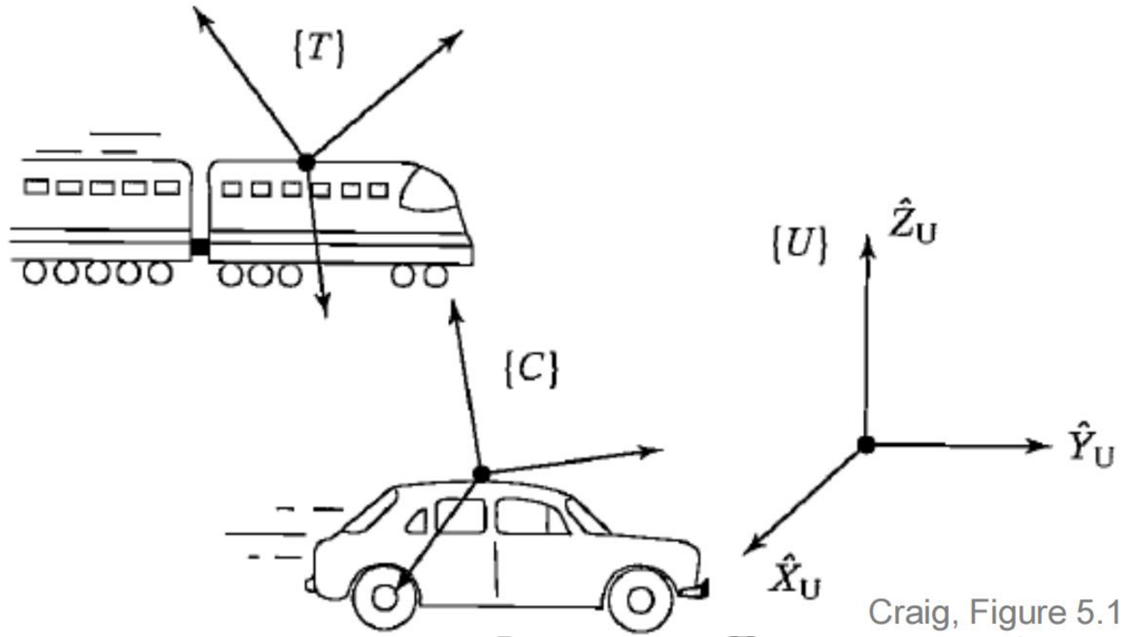
当两个坐标系是同一个坐标系时

$$\upsilon_C = \,^{U}V_{C\,ORG}$$

坐标系 {C} 的原点相对于世界参考坐标系的速度。

时变位置和姿态的符号表示

- 例



Craig, Figure 5.1

$$^{U}V_T = 100\hat{\imath}$$

$$^{U}V_C = 30\hat{\imath}$$

$$^{U}\left(\frac{d}{dt}\,^{U}P_{C\,ORG}\right) = \,^{U}V_{C\,ORG} = v_C = 30\hat{\imath}$$

$$^{C}(^{U}V_{T\,ORG}) = \,^{C}v_T = \,^{C}_{U}R(v_T) = \,^{C}_{U}R(100\hat{\imath}) = \,^{U}_{C}R^{-1}100\hat{\imath}$$

$$^{C}(\,^{T}V_{C\,ORG}) = \,^{C}_{T}R(\,^{T}(\,^{T}V_{C\,ORG})) = \,^{C}_{T}R(\,^{T}V_{C\,ORG})$$
$$= \,^{C}_{U}R\,^{U}_{T}R(-70\hat{\imath}) = -\,^{U}_{C}R^{-1}\,^{U}_{T}R70\hat{\imath}$$

时变位置和姿态的符号表示

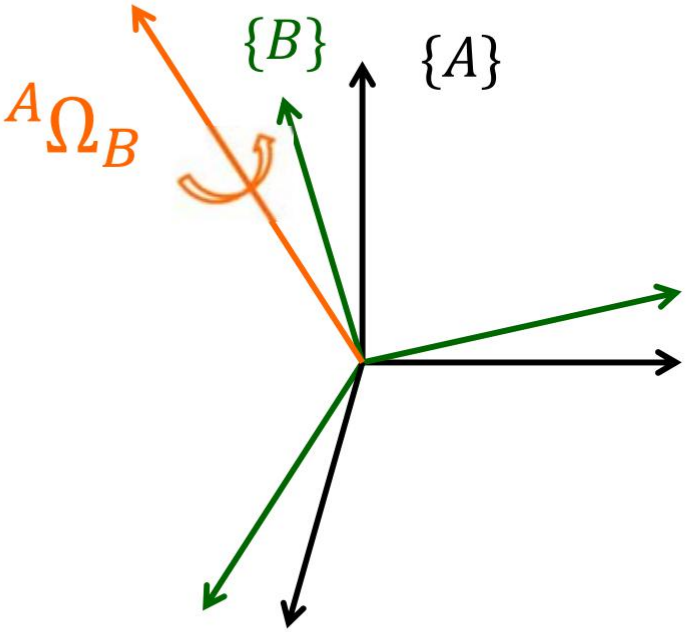
- 角速度矢量 ${}^{A}\Omega_{B}$
  - 坐标系 {B} 相对于坐标系 {A} 的转动
  - ${}^{A}\Omega_{B}$ 的方向：瞬时的旋转轴
  - ${}^{A}\Omega_{B}$ 的大小：旋转的速度

${}^{C}({}^{A}\Omega_{B})$

在坐标系 {C} 的表达

$\omega_{C} = {}^{U}\Omega_{C}$

坐标系 {C} 相对于世界参考坐标系 {U} 的角速度。



刚体的运动

➢ 线速度矢量

$$\overrightarrow{r_A}=x_A\hat{I}+y_A\hat{J}$$

$$=\overrightarrow{r_B}+\overrightarrow{r_{A/B}}$$

$$=(x_B\hat{I}+y_B\hat{J})+(x_{A/B}\hat{I}+y_{A/B}\hat{J})$$

$$=\overrightarrow{r_B}+\overrightarrow{r_{A/B}}$$

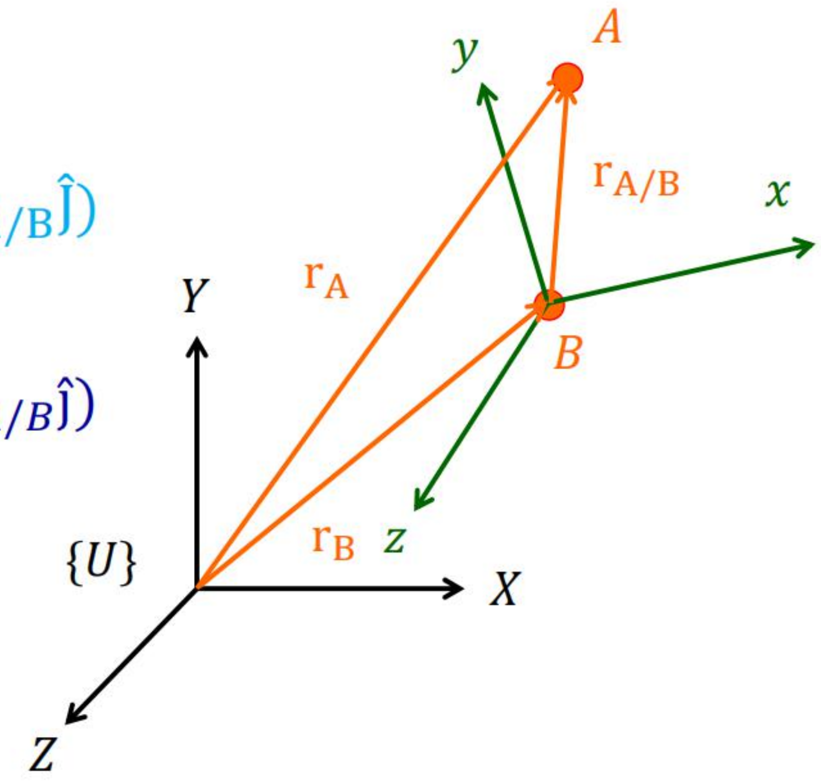
$$=(x_B\hat{I}+y_B\hat{J})+(x_{A/B}\hat{\imath}+y_{A/B}\hat{\jmath})$$

↓ diff.

$$\overrightarrow{v_A}=\dot{\overrightarrow{r_A}}=\dot{x}_A\hat{I}+\dot{y}_A\hat{J}$$

$$=\dot{\overrightarrow{r_B}}+\dot{\overrightarrow{r_{A/B}}}$$

$$=(\dot{x}_B\hat{I}+\dot{y}_B\hat{J})+(\dot{x}_{A/B}\hat{I}+\dot{y}_{A/B}\hat{J})$$



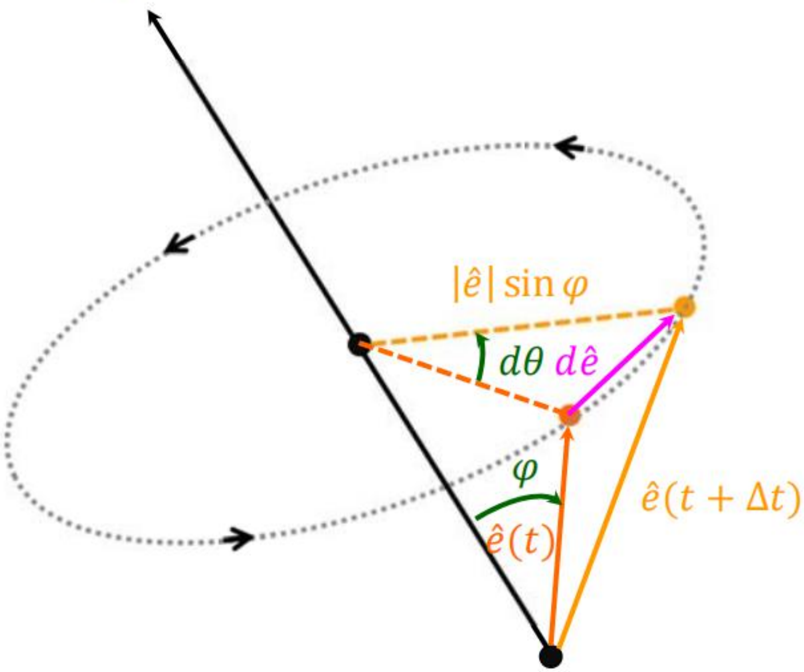
刚体的运动

- $\overrightarrow{v_A} = \overrightarrow{\dot{r}_B} + \overrightarrow{\dot{r}_{A/B}}$

$$= (\dot{x}_B\hat{\imath} + \dot{y}_B\hat{\jmath}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \underline{(x_{A/B}\dot{\hat{\imath}} + y_{A/B}\dot{\hat{\jmath}})}$$

$$= x_{A/B}\,(\vec{\omega}\times\hat{\imath}) + y_{A/B}\,(\vec{\omega}\times\hat{\jmath})$$

$\vec{\omega}$ $\quad |\vec{\omega}| = \dot{\theta}$



Magnitude:

$$|d\hat{e}| = |\hat{e}|sin\varphi d\theta$$

$$|\dot{\hat{e}}| = |\hat{e}|sin\varphi\dot{\theta} = |\hat{e}||\vec{\omega}|sin\varphi$$

Direction:

$$d\hat{e} \perp \hat{e}$$

$$d\hat{e} \perp \vec{\omega}$$

⇨ $\dot{\hat{e}} = \vec{\omega}\times\hat{e}$

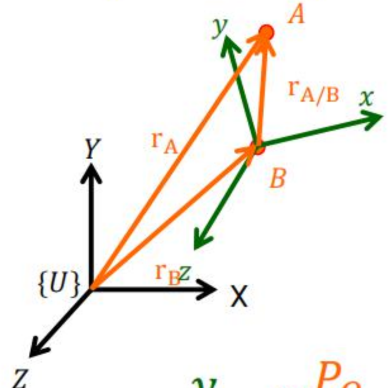
刚体的运动

$$\overrightarrow{v_A} = (\dot{x}_B \hat{I} + \dot{y}_B \hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{\imath} + y_{A/B}\hat{\jmath})$$

$$= (\dot{x}_B \hat{I} + \dot{y}_B \hat{J}) + (\dot{x}_{A/B}\hat{\imath} + \dot{y}_{A/B}\hat{\jmath}) + \vec{\omega} \times (x_{A/B}\hat{I} + y_{A/B}\hat{J})$$

$$\Rightarrow \quad \overrightarrow{v_A} = \overrightarrow{v_B} + \underline{\overrightarrow{v_{rel}}} + \vec{\omega} \times \overrightarrow{r_{A/B}}$$

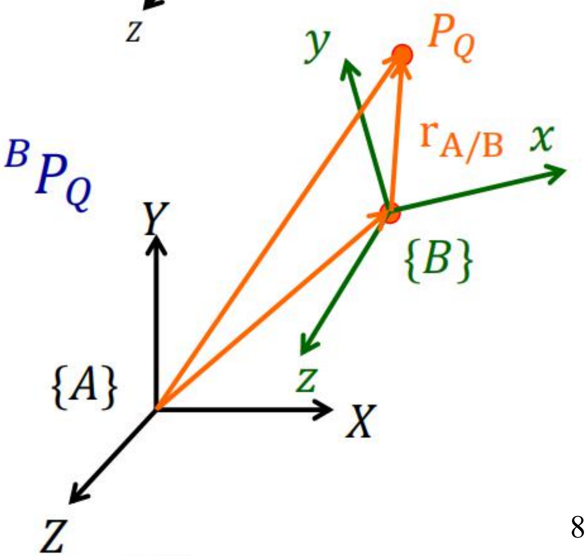
"relative" velocity



- Thus,

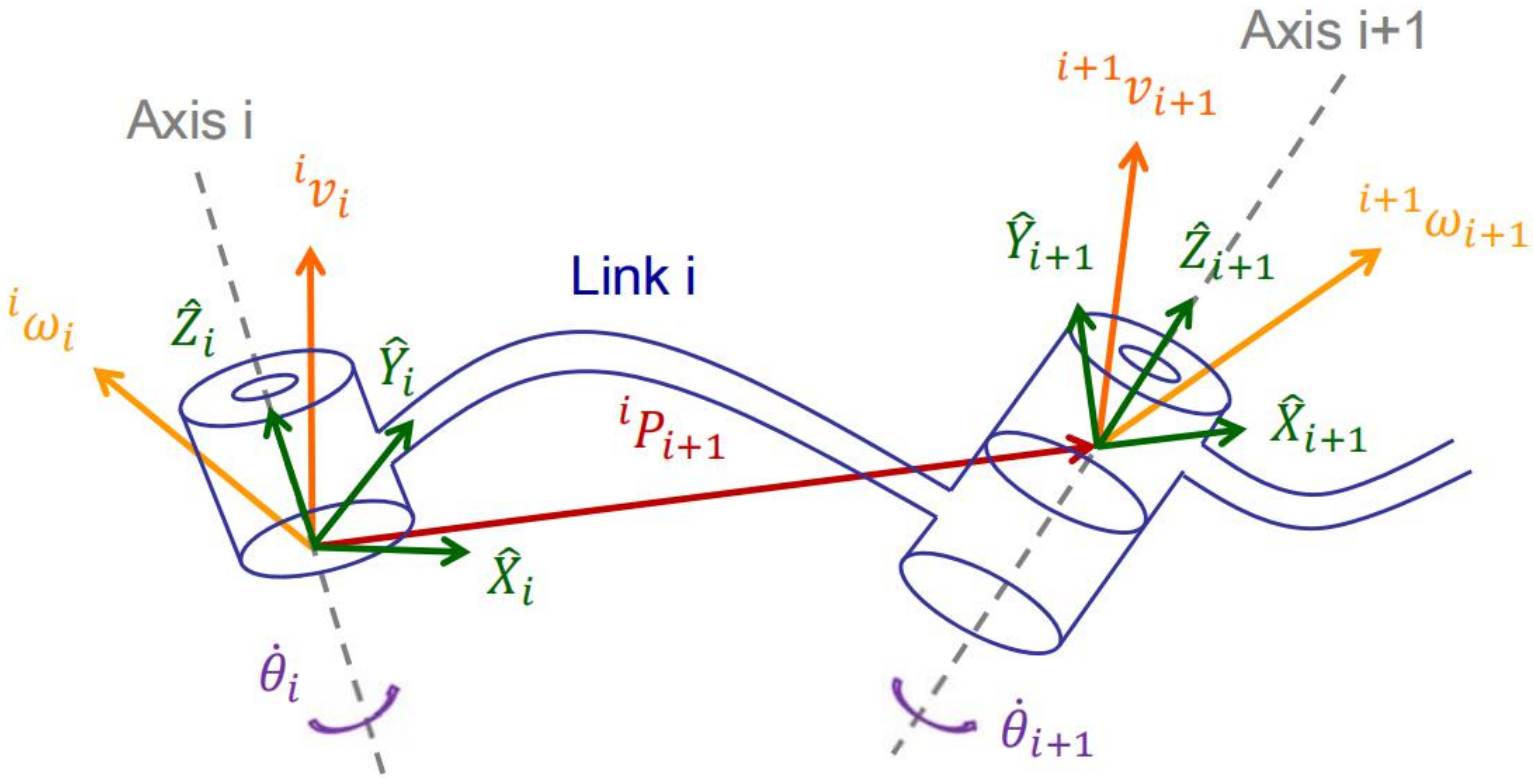
$$^{A}V_{Q} = {}^{A}V_{B\,ORG} + {}^{A}_{B}R\,\underline{^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}$$

"relative" velocity



速度在连杆之间的传递

➢ 策略：已知连杆i在坐标系{i}中的线速度和角速度，找到它们和相邻连杆之间的速度和角速度的关系。



速度在连杆之间的传递

- 旋转关节（连杆i+1）
  - 角速度的传递

$$^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$\downarrow {}^{i+1}_{i}R$

$$\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

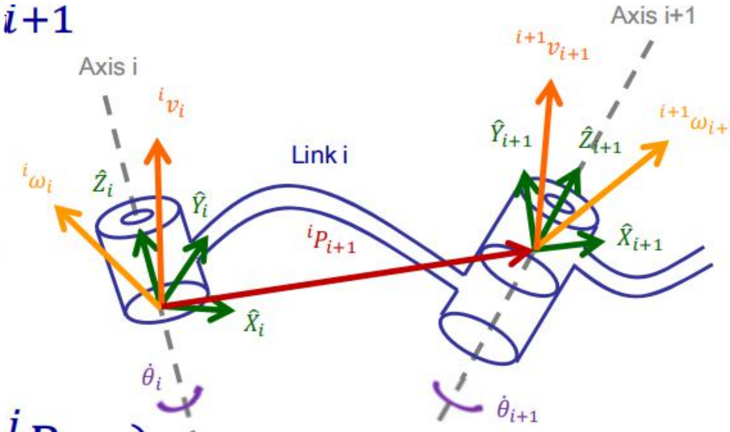
$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

  - 线速度的传递

$$^{i}v_{i+1} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}$$

$\downarrow {}^{i+1}_{i}R$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1})$$



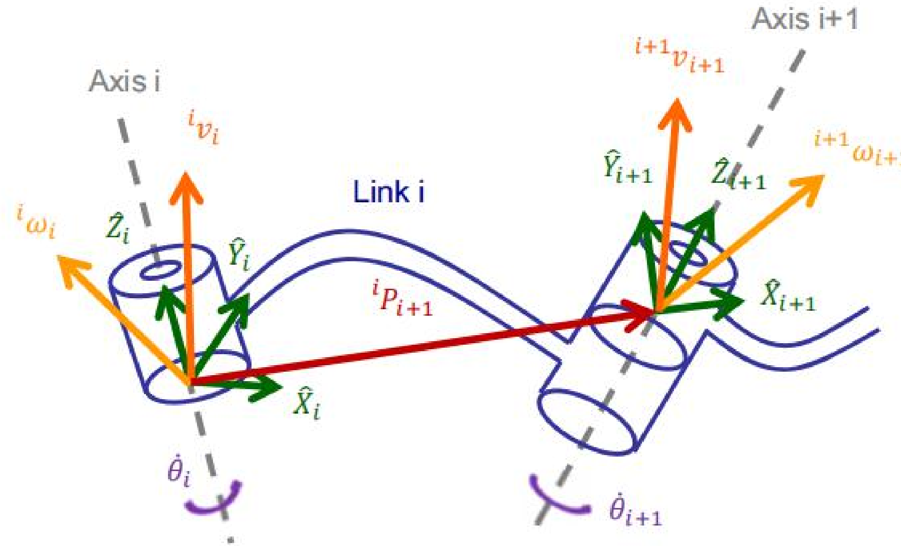
速度在连杆之间的传递

- 平动关节（连杆i+1）
  - 角速度的传递

$${}^{i}\omega_{i+1} = {}^{i}\omega_{i}$$

$$\downarrow {}^{i+1}_{i}R$$

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_{i}$$



  - 线速度的传递

$${}^{i}v_{i+1} = \left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right) + {}_{i+1}^{i}R\,\underline{\dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\downarrow {}^{i+1}_{i}R \qquad \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

$${}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

雅可比矩阵

➢ 多维形式的微分

$$y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6)$$

$$\vdots$$

$$y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6)$$

⇨ $Y = F(X)$

雅可比矩阵

➢ 计算 y_i 的微分关于 x_i 的微分

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

Jacobian, "linear transformation"

➡ $$\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

Function of X, if f_i is nonlinear

➡ $$\dot{Y} = J(X)\dot{X}$$

雅可比矩阵

- 在机器人中
  - 可以将机械臂末端的笛卡尔速度和关节速度关联起来

$$^{0}\boldsymbol{\nu} = \begin{bmatrix} ^{0}v \\ ^{0}\omega \end{bmatrix} = {}^{0}J(\Theta)\dot{\Theta}$$

3 × 1 ： 平面运动

6 × 1 ： 空间运动

- 改变雅可比矩阵的参考坐标系（空间运动）

$$^{B}\boldsymbol{\nu} = \begin{bmatrix} ^{B}v \\ ^{B}\omega \end{bmatrix} = {}^{B}J(\Theta)\dot{\Theta}$$

$$^{A}\boldsymbol{\nu} = \begin{bmatrix} ^{A}v \\ ^{A}\omega \end{bmatrix} = {}^{A}J(\Theta)\dot{\Theta} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} ^{B}v \\ ^{B}\omega \end{bmatrix}$$

$$\Rightarrow \quad {}^{A}J(\Theta) = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta)$$

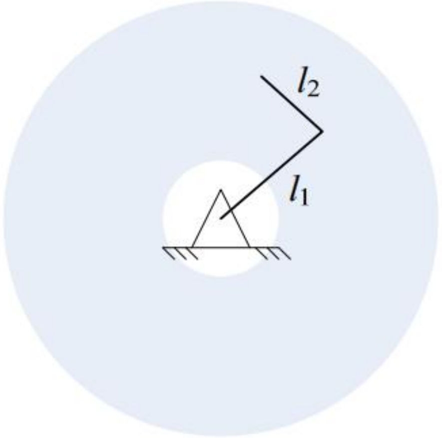
雅可比矩阵

- 可逆性

$$\dot{\Theta} = J^{-1}(\Theta)\boldsymbol{v}$$

- ◆ 奇异性：当雅可比矩阵是不可逆的时候
  - ● 工作空间的奇异位形

    例：当机械臂完全伸直或者折叠
  - ● 工作空间内部的奇异位形
- ◆ 当一个机械臂处于奇异位形时
  - ● 损失一个或者多个自由度



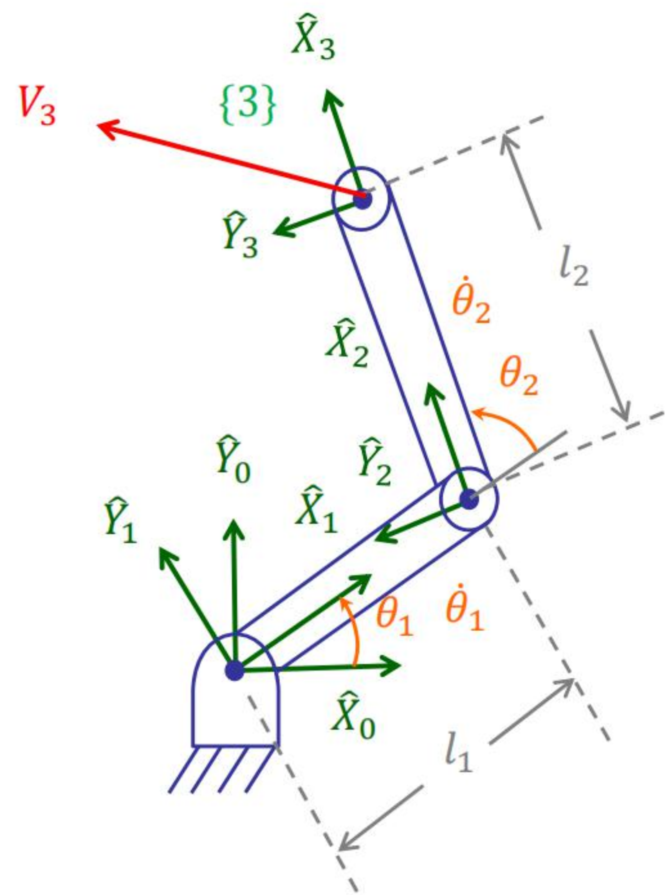
例：一个RR机械臂

➢ 方法一：速度在连杆之间传递

$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



例：一个RR机械臂

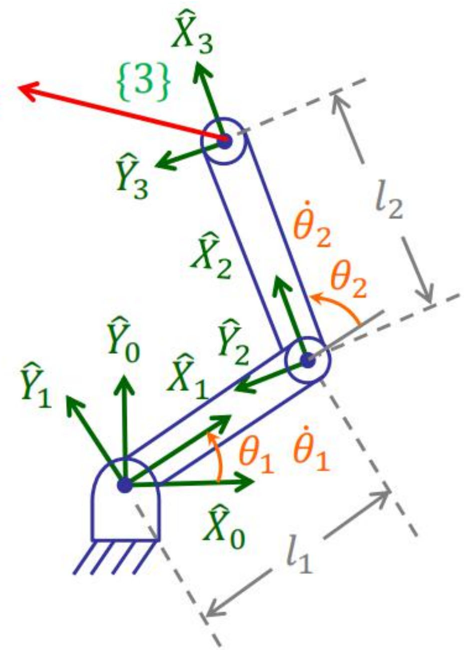
◆ 连杆传递

$$ {}^{1}\omega_1 = {}^{1}_{0}R\ {}^{0}\cancel{\omega_0} + \dot{\theta}_1\ {}^{1}\hat{Z}_1 = \dot{\theta}_1\ {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ {}^{1}v_1 = {}^{1}_{0}R(\ {}^{0}\cancel{v_0} + {}^{0}\cancel{\omega_0} \times {}^{0}\cancel{P_1}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{2}\omega_2 = {}^{2}_{1}R\ {}^{1}\omega_1 + \dot{\theta}_2\ {}^{2}\hat{Z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ {}^{2}v_2 = {}^{2}_{1}R(\ {}^{1}\cancel{v_1} + {}^{1}\omega_1 \times {}^{1}P_2) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} $$



例：一个RR机械臂

$$^{3}\omega_3 = {}^{2}\omega_2$$

$$^{3}v_3 = {}^{3}_{2}R\left({}^{2}v_2 + {}^{2}\omega_2 \times {}^{2}P_3\right)$$

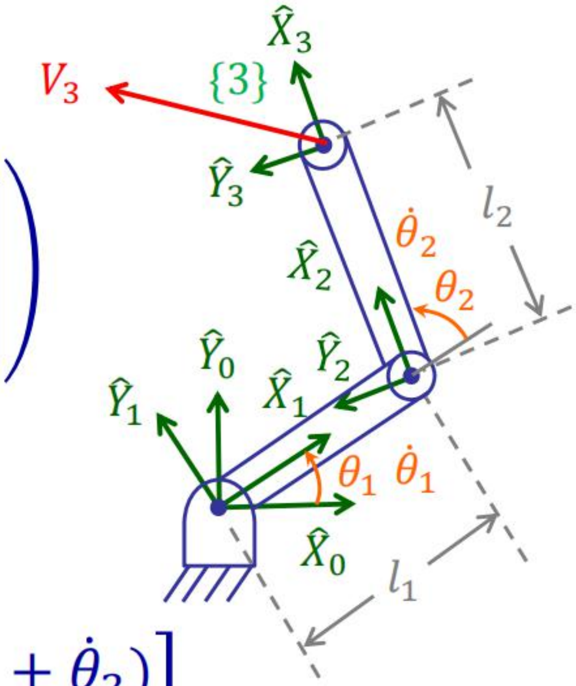
$$= I\left(\begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$^{0}v_3 = {}^{0}_{3}R\ {}^{3}v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

$$= {}^{0}_{1}R\,{}^{1}_{2}R\,{}^{2}_{3}R$$

$$= \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

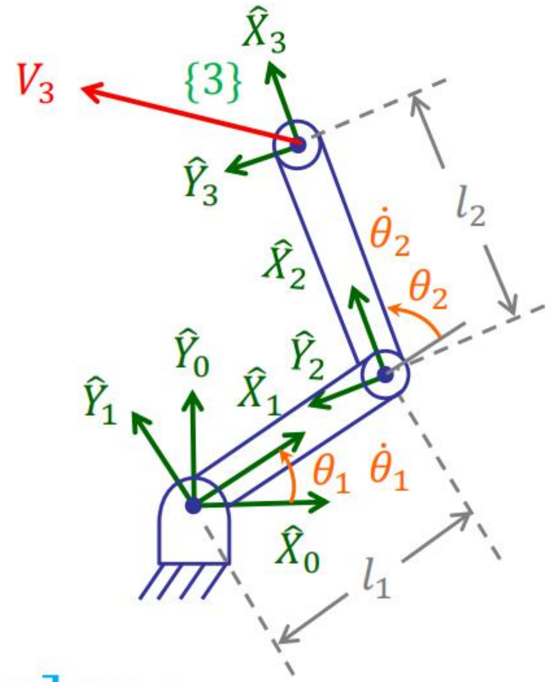


例：一个RR机械臂

- 因此

$$^{3}\boldsymbol{v}=\begin{bmatrix}v_x\\v_y\\\omega\end{bmatrix}=\begin{bmatrix}l_1s_2 & 0\\l_1c_2+l_2 & l_2\\1 & 1\end{bmatrix}\begin{bmatrix}\dot\theta_1\\\dot\theta_2\end{bmatrix}$$

$$= \; ^{3}J(\Theta)\dot\Theta$$



$$det\begin{vmatrix}l_1s_2 & 0\\l_1c_2+l_2 & l_2\end{vmatrix}=l_1l_2s_2=0$$

⇨ $\theta_2 = 0 \; or \; 180$

$$^{0}\boldsymbol{v}=\begin{bmatrix}v_x\\v_y\\\omega\end{bmatrix}=\begin{bmatrix}-l_1s_1-l_2s_{12} & -l_2s_{12}\\l_1c_1+l_2c_{12} & l_2c_{12}\\1 & 1\end{bmatrix}\begin{bmatrix}\dot\theta_1\\\dot\theta_2\end{bmatrix}$$

$$= \; ^{0}J(\Theta)\dot\Theta$$

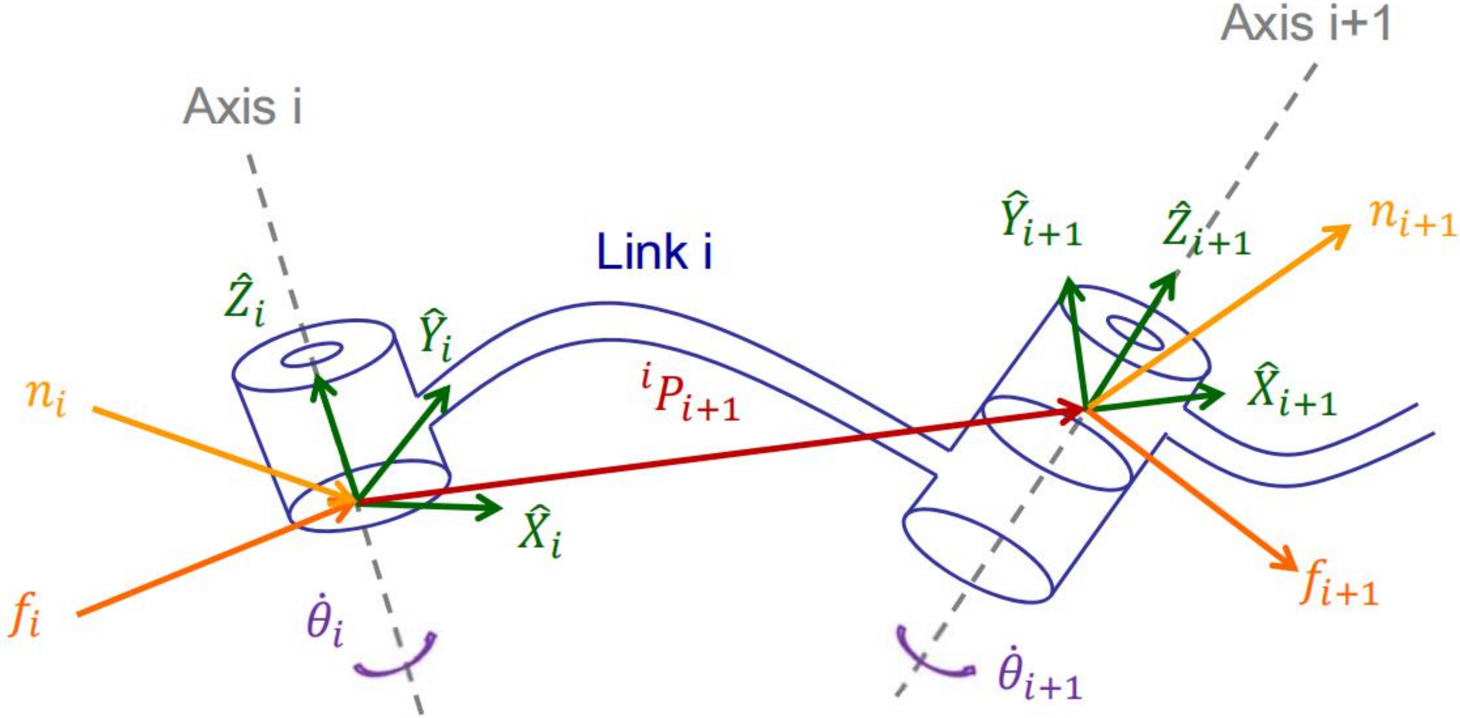
机械臂的静态力

- 当考虑静态力的时候
  - 锁定所有的关节
  - 写出力和力矩的关系
  - 计算静态力矩（忽略重力）

机械臂的静态力

- $^{i}f_i = {}^{i}f_{i+1}$ $\quad\xrightarrow{{}^{i}_{i+1}R}\quad$ $^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$

  $^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1}$ $\quad\xrightarrow{{}^{i}_{i+1}R}\quad$ $^{i}n_i = {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$



机械臂的静态力

- 关节力矩需要维持关节的静态平衡
  - 旋转关节

$$\tau_i = {}^i n_i^T \, {}^i\hat{Z}_i$$

  - 平动关节

$$\tau_i = {}^i f_i^T \, {}^i\hat{Z}_i$$

例：一个RR机械臂

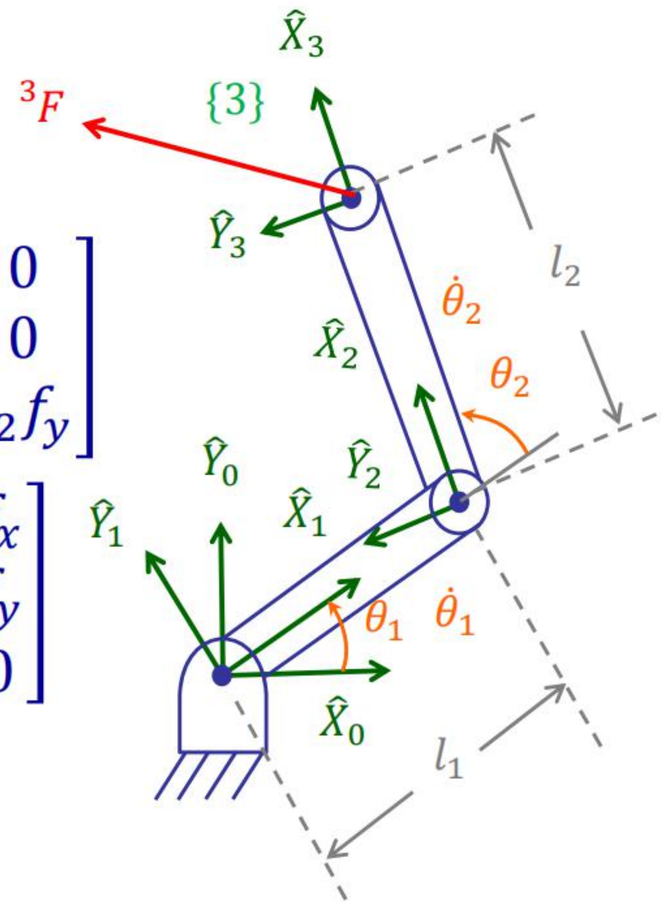
◆ 力在连杆之间的传递

$$^{2}f_2={}^{2}_{3}R\ {}^{3}f_3 = I\ {}^{3}F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$$^{2}n_2 = {}^{2}_{3}R\ {}^{3}n_3 + {}^{2}P_3 \times {}^{2}f_2 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}$$

$$^{1}f_1 = {}^{1}_{2}R\ {}^{2}f_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix}$$



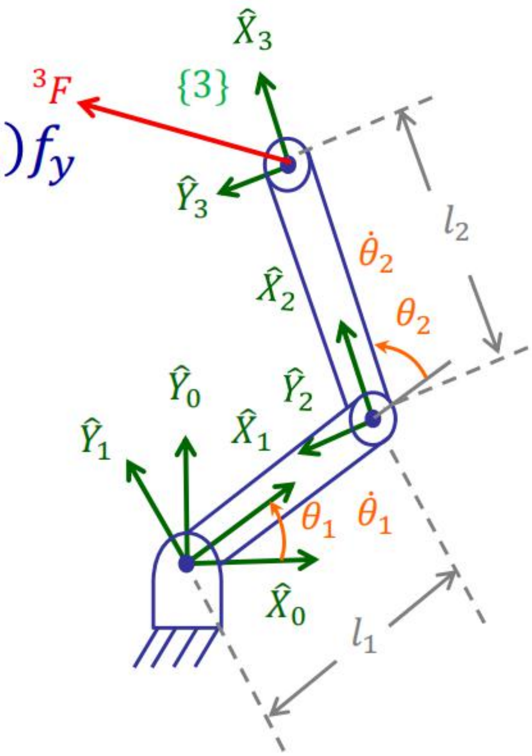
例：一个RR机械臂

$$ {}^{1}n_{1} = {}^{1}_{2}R\ {}^{2}n_{2} + {}^{1}P_{2} \times {}^{1}f_{1} = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} $$

- 因此

$$ \tau_1 = {}^{1}n_1^T\ {}^{1}\widehat{Z_1} = l_1 s_2 f_x + (l_1 c_2 + l_2) f_y $$

$$ \tau_2 = {}^{2}n_2^T\ {}^{2}\widehat{Z_2} = l_2 f_y $$



$$ \Rightarrow \tau = \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} $$

力域的雅可比矩阵

- 虚拟功原理

$$F \cdot \delta \mathcal{X} = \Gamma \cdot \delta \Theta$$

$$F^T \delta \mathcal{X} = F^T J \delta \Theta = \Gamma^T \delta \Theta$$

$$\Gamma = J^T F$$

相对于坐标系 {0}

$$\Rightarrow \quad \Gamma = {}^0J^T \, {}^0F$$

不使用逆向运动学实现笛卡尔力矩到关节力矩的变换

速度和静力的笛卡尔变换

- 广义的速度和力的表达式

$$\boldsymbol{\nu} = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}$$

速度和静力的笛卡尔变换

- 广义的速度和力的表达式

$$\boldsymbol{\nu} = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}$$

- 坐标系的变换

速度和静力的笛卡尔变换

- 广义的速度和力的表达式

$$\boldsymbol{\nu} = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}$$

- 坐标系的变换

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R(\ {}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1})$$

速度和静力的笛卡尔变换

- 广义的速度和力的表达式

$$\boldsymbol{\nu} = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad \boldsymbol{\mathcal{F}} = \begin{bmatrix} F \\ N \end{bmatrix}$$

- 坐标系的变换

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R(\ {}^{i}v_{i} + \ {}^{i}\omega_{i} \times \ {}^{i}P_{i+1})$$

$$\Big\downarrow\ i = B, i+1 = A, \dot{\theta} = 0$$

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$$\begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,ORG} \times {}^{A}_{B}R \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix}$$

$$P\times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$

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$$^{A}\boldsymbol{\nu}_A = {}^{A}_{B}T_v\ {}^{B}\boldsymbol{\nu}_B \qquad P\times = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}$$

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↓ "inverse"

$$\begin{bmatrix} {}^{B}v_{B} \\ {}^{B}\omega_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & -{}^{B}_{A}R\, {}^{A}P_{BORG}\times \\ 0 & {}^{B}_{A}R \end{bmatrix} \begin{bmatrix} {}^{A}v_{A} \\ {}^{A}\omega_{A} \end{bmatrix}$$

$$ {}^{B}\boldsymbol{v}_{B} = {}^{B}_{A}T_{v}\, {}^{A}\boldsymbol{v}_{A}$$

- 相似的

$$\begin{bmatrix} {}^{A}F_{A} \\ {}^{A}N_{A} \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ {}^{A}P_{BORG}\times {}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}F_{B} \\ {}^{B}N_{B} \end{bmatrix}$$

$$ {}^{A}\boldsymbol{\mathcal{F}}_{A} = {}^{A}_{B}T_{f}\, {}^{B}\boldsymbol{\mathcal{F}}_{B}$$

⇨ $${}^{A}_{B}T_{f} = {}^{A}_{B}T_{v}{}^{T}$$

谢 谢！